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CONES IN LATTICE ORDERED LOOPS

V. B. V. N. PRASAD[1] & J. VENKATESWARA RAO[2]

[1]Associate Professor, Department of Mathematics, Vignan Degree & P. G College, Guntur, Andhra Pradesh, India

[2]Professor, Department of Mathematics, Mekelle University, Mekelle, Ethiopia

ABSTRACT

This manuscript illustrates the significance of cones in lattice ordered loops. Here various properties of cones in lattice ordered loops obtained with additive notation. A prerequisite identified for lattice ordered loop to develop into positive and negative cones and an equivalent condition for positive, negative cones and left nucleus in lattice ordered loops.

KEYWORDS: Loops, Quasi Groups, Lattices and their Related Structures, Lattices

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1. INTRODUCTION

Garrett Birkhoff (1942) firstly initiated the notion of lattice ordered groups. Then Bruck (1944) contributed various results in the theory of quasigroups. Zelinski (1948) described about ordered loops. The concept of non associative number theory was thoroughly studied by Evans (1957). Bruck (1963) explained about what is a loop? Various crucial properties of lattice ordered groups were established by Garrett Birkhoff in 1964 and 1967. Evans (1970) described about lattice ordered loops and quasigroups. Richard Hubert Bruck (1971) made a survey of binary systems. In the recent past Hala (1990) made a description on quasigroups and loops.

In this manuscript, we assume that L is a lattice ordered loop. Further, we furnish definitions, examples and some properties of cones in lattice ordered loops. We initiate the concepts of positive and negative cones, definitions and examples of quasigroups, loop, partially ordered loop, lattice ordered loop, positive element, directed loop, the left nucleus of a loop and some important lemmas and theorems on positive and negative cones. The properties and the definitions, examples are in additive notation.

2. CONES IN LATTICE ORDERED LOOPS

Definition (2.1): A quasigroup (S, +) is an algebra, where S is a non empty set, with a binary addition +, in which any two of the three terms of the equation a + b = c uniquely determine the third.

Example (2.1): Let $\mathbb{Z}$ be the set of integers together with the operation usual substraction '$-$.' Clearly a – c and a + b are unique solutions of a – x = c and y – b = a respectively. So $(\mathbb{Z}, -)$ is a quasigroup.

Note (2.1): The quasigroup in the above example is not a group because it does not satisfy the associative property. In fact 2 – (3 – 1) = 2 – 2 = 0 and (2 – 3) – 1 = $-$ 1 – 1 = $-$ 2.

So every quasigroup is not a group.

Definition (2.2): A loop is a quasigroup (S, +) with two sided identity 0.

That is 0 + x = x + 0 = x for all x in S.

Note (2.2): It follows that the identity element 0 is unique and that every element of S has unique left and right inverse.

Example (2.2): Every group is a loop, because a + x = b if and only if x = (−a) + b and y + a=b if and only if y = b + (−a).

Note (2.3): In a loop x /x = x − x = 0 and x \x = − x + x = 0 for any x, where / is a right division and \ is a left division in a loop.

Definition(2.3): A system (S, +, ≤), where S is a nonempty set, + is a binary operation on S and ≤ is a binary relation on S satisfying

- (S, +) is a loop
- (S, ≤) is a partially ordered set
- The translations x $\mapsto$ a + x and x $\mapsto$ x + b are ordered automorphisms of S, is called a partially ordered loop (briefly P.O. loop).

Example (2.3): (Z, +,≤) is a partially ordered loop.

Definition (2.4): A lattice ordered loop L is a partially ordered loop in which the partial order is a lattice order.

Definition (2.5): Let L is a lattice ordered loop. An element x of L is called positive element if x ≥ 0.

Definition (2.6): In a lattice ordered loop L, a positive element which covers 0 is called an atom.

Lemma (2.1): In any lattice ordered loop L, for all a, b, c ∈ L, a ≤ b ⇔ a – c ≤ b - c and also c - b ≤ c - a and hence -b ≤ -a, a-b ≤ 0.

Proof: Suppose that for all a, b, c ∈ L, a ≤ b. Then a ≤ b ⇔ c + (a - c) ≤ c + (b - c) ⇔ a - c ≤ b - c

And a ≤ b ⇔ a + (c - a) + b + (c - a) ⇔ b + (c - b) ≤ b + (c - a) ⇔ c - b ≤ c – a. This completes the proof.

Lemma (2.2): Except in the trivial case of L = {0}, a lattice ordered loop L cannot have universal bounds.

Proof: It is clear by definitions of positive and negative cones in a lattice ordered loop L.

Definition (2.7): Let L is a lattice ordered loop. The set P = {x ∈ L / x ≥ 0} is called as the positive cone of L. Also we write P = {x ∈ L / 0 – x ∈ *P*}.

Theorem (2.1): Let L is a lattice ordered loop. Positive cone of L is P = {x ∈ L / x ≥ 0}. Then P satisfies the following:

- P ∩ -P = {0}

- $P + P \subseteq P$
- $-a + P + a = P \ \forall \ a \in L$.

Proof

(i) Let $x \in P \cap -P$. Then $x \in P$ and $x \in -P$. So, $x \in P$ and $-x \in P$. This implies $x \geq 0$ and $-x \geq 0$.

Which implies that $x \geq 0$ and $x \leq 0$. Hence, $x = 0$. Therefore $P \cap - P \subseteq \{0\} \rightarrow (1)$.

Also consider $0 \in P$ and $0 \in -P$. Then $0 \in P \cap -P$. This implies $\{0\} \subseteq P \cap - P \rightarrow (2)$.

Hence from (1) and (2), $P \cap - P = \{0\}$.

(ii) Let $z \in P + P$. Then $z = x + y$, $x \in P$ and $y \in P$. So, $x \geq 0$ and $y \geq 0$.

Which implies, $x + y \geq 0 + y = y \geq 0$. Hence, $x + y \in P$. Therefore $P + P \subseteq P$.

(iii) Let $a \in L$ and $x \in P$. So, $x \geq 0$. This leads to $-a + x + a \geq -a + 0 + a = 0$. Which implies, $-a + x + a \in P$.

Therefore, $-a + P + a \subseteq P \rightarrow (1)$.

Also let $x \in P$. Then, $x \geq 0$. Consider $x = -a + a + x - a + a = -a + (a + x - a) + a$.

Now $x \geq 0 \Rightarrow a + x - a \geq a + 0 - a = 0$. Therefore $a + x - a \in P$. Hence $x \in -a + P + a$.

Thus $P \subseteq -a + P + a \rightarrow (2)$. From (1) and (2), $-a + P + a = P$ for all $a \in L$.

Corollary (2.1): If $x \leq y$ then $a + x + b \leq a + y + b$ for all a, b.

Proof: Let $x \leq y$ and $a, b \in L$. So $y - x \in P$. Consider $(a + y + b) - (a + x + b) = a + y + b - b - x - a$

$= a + y - x - a \in a + P - a = P + a - a = P$. Therefore, $a + x + b \leq a + y + b$.

Theorem (2.2): Any lattice ordered loop L is determined to within isomorphism by its positive cone

$P = L^+$, and we have $a \leq b$, $b - a \in P$ and $-a + b \in P$ are equivalent.

Moreover (i) $0 \in P$; (ii) if $x, y \in P$, then $x + y \in P$; (iii) if $x, y \in P$ and $x + y = 0$ then, $x = y = 0$ and (iv) For all $a \in L$, $a + P = P + a$.

Proof: Suppose $a \leq b \Rightarrow b - a \in P$. Suppose $b - a \in P \Rightarrow b - a \geq 0$.

This implies, $-a + b - a + a \geq - a + 0 + a = 0$. So, $- a + b \geq 0$. This leads to $- a + b \in P$.

Suppose $- a + b \in P$. Then, $- a + b \geq 0$. So $a - a + b \geq a + 0$. Which implies, $b \geq a$ and so $a \leq b$.

(i) $a \leq a$ implies, $a - a \in P$. So $0 \in P$.

(ii) If $x, y \in P$, $x \geq 0$, $y \geq 0$. Then, $x + y \geq 0$ and hence, $x + y \in P$.

(iii) $x \in P$ implies, $x \geq 0$. Then, $x + y \geq 0 + y = y \geq 0$. So we have $0 \geq y \geq 0$.

Which leads to $y = 0$. Similarly we can see that $x = 0$.

(iv) $a + P = \{a + x / x \in P$, that is $x \geq 0\} = \{x \geq a\}$.

$P + a = \{x + a / x \in P$, that is $x \geq 0\} = \{x \geq a\}$.

(or) If P is a positive cone $\Leftrightarrow -a + P + a = P \Leftrightarrow a - a + P + a = a + P \Leftrightarrow P + a = a + P$.

Lemma (2.3): In a lattice ordered loop L with positive cone P, for all $a, x \in L$, we have the following:

(i) $x \in P$, $x + P = P$ implies $x = 0$.

(ii) $a + (x + P) \subseteq (a + x) + P$.

(iii) $(x + P) - a \subseteq (x - a) + P$.

Proof

(i) Suppose that $x + P = P$. This implies, $- x \in P$. So, $x \in -P$. Hence, $x = 0$.

(ii) For $p \in P$, consider, $a + (x + p) \geq a + x$. This implies, $[a + (x + p)] = a + x \in P$ and hence, the proof follows.

(iii) For $p \in P$, Consider, $(x + p) - a \geq x - a$. This implies, $[(x + p) -a] -(x - a) \in P$ and hence, the proof follows.

Theorem (2.3): In a lattice ordered loop L, the positive cone P has the following properties:

For all $a, x, b \in L$,

(i) $P \cap -P = \{0\}$.

(ii) $(a + P) + P = a + P$.

(iii) $a +(x + P) = (a + x) + P$.

(iv) $(x + P) -a = (x - a) + P$.

(v) $a \leq b$ if and only if $b - a \in P$.

Conversely if L is a lattice ordered loop and $P \subseteq L$, satisfying (i) to (iv), then by defining $a \leq b$ if and only if $(b - a) \in P$, L becomes a lattice ordered loop in which P is precisely the positive cone.

Proof

(i) By definitions of P and $-P$, $P \cap -P = \{0\}$.

(ii) Clearly $a + P \subseteq (a + P) + P$. For $p, q \in P$, $[(a + P) + q] \geq a$. This implies, $[(a + P) + q] - a \in P$.

(iii) In view of lemma (2.3) (ii), it suffices to show that $(a + x) + P \subseteq a + (x + P)$.

Now consider $[(a + x) + P] - a \subseteq [(a + x) - a] + P$

$= x + P$, by using lemma (2.3)(iii).

(iv) $[(x - a) + P] + a \subseteq [(x - a) + a] + P$, by lemma (2.3)(ii)

$= x + P$ and the proof follows by lemma (2.3)(iii)

(v) Follows from lemma (2.1).

Conversely let P satisfies (i) to (v).

Then clearly the condition (ii) leads to $a \leq b \Leftrightarrow b + P \subseteq a + P$, which together with lemma (2.3) (i) show that '≤' is a partial order relation. Clearly the map $x \rightarrow x + a$ is a bijection.

Now $x \leq y \Leftrightarrow y - x \in P \Leftrightarrow y \in x + P \Leftrightarrow y + a \in (x + P) + a = (x + a) + P \Leftrightarrow x + a \leq y + a$.

This completes the proof.

Lemma (2.4): In any lattice ordered loop L the positive cone p is invariant under all inner-automorphisms, namely, $x \rightarrow (- a + x) + a$ and $x \rightarrow (-a) + (x + a)$.

Proof: Follows from the definition of lattice ordered loop.

Definition (2.8): A directed loop is a po-loop (S, +) having the More-Smith property: (1) Given a, b$\in S$, there exists, c$\in S$ such that $a, b \leq c$

Lemma (2.5): A po-loop (S, +) is directed if and only if every element is expressed as the difference of two positive elements in the sense that $x \in S \Rightarrow x = a - b$, where a and b are two positive elements in

(S, +).

Proof: Let a$\in S$. By our assumption there exists c $\in S$ such that $a \leq c$ and $0 \leq c$, since $0 \in S$.

So $a \leq c$ implies, $- a + a \leq - a + c$. Which implies $-a + c \geq 0$ and $c \geq o$.

Now $a = c - (-a + c)$, where $c \in S^+$ and $-a + c \in S$.

Conversely, let $a = a' - a''$ and $b = b' - b''$, where a', a'', b', b'' are positive.

Since $-a'' \leq 0$, $a \leq a' - 0 = a' + 0 \leq a' + b'$ (since $b' \geq 0$).

So $b \leq b' - 0 = b' = 0 + b' \leq a' + b'$. Thus $a' + b'$ is an upper bound to a and b. Hence the proof.

Note (2.4): For our convenience positive, negative cones are denoted by L^+, L^- instead of P and -P in a lattice ordered loop L.

Definition (2.9): Let L is a lattice ordered loop. The Positive Cone L^+ of L is the set of all elements

$x \geq 0$. That is $L^+ = \{x \in L \,/\, x \geq 0\}$.

Definition (2.10): Let L is a lattice ordered loop. The Negative Cone L^- of L is the set of all elements

$x \leq 0$. That is $L^- = \{x \in L \,/\, x \leq 0\}$.

Note (2.5): If x is a positive element in L, then $x \geq a_1$ where a_1 is some positive atom. If $x > a_1$, then $x - a_1$ is positive and hence $x - a_1 \geq a_2$ where a_2 is some positive atom. Continuing in this way, we obtain the descending chain condition $\{\ldots.[(x - a_1) - a_2]\ldots\} - a_{t-1} = a_t$, where the a_t are positive atoms.

Then $x = \{\ldots[(a_t + a_{t-1}) + a_{t-2}] + \ldots\ldots\} + a_1$. Hence, $L^+ - \{0\}$ is generated under addition by the positive atoms. Similarly we can see that $L^- - \{0\}$ is generated under addition by the negative atoms.

Definition (2.11): The left nucleus of a loop is the set of all elements x such that $(x + y) + z = x + (y + z)$ for all y, z.

Definition (2.12): Let S be a non empty set and + is a binary operation on S. Then the algebraic structure (S, +) is called a groupoid if $a + b \in S$ for all $a, b \in S$.

Example (2.4): Let Z be the set of integers and addition (+) is a binary operation on Z then (Z, +) is a groupoid, because $a + b \in Z$ for all $a, b \in Z$.

Lemma (2.6): If a is a positive atom, then a generates a semi group under addition.

Proof: Let A be the groupoid generated by 'a' under addition.

We note first of all that if x is any element in A and 'b' is any positive atom different from 'a', then by ($x \wedge y = 0$ and $x \wedge z = 0 \Rightarrow x \wedge (y + z) = 0$), $x \wedge b = 0$. Let x, y be any elements in A.

Then a + x covers x and (a + x) + y covers x + y. Hence $\{(a + x) + y\} - (x + y)$ covers 0.

That is, $(a + x) + y = b + (x + y)$ where b is positive atom. If $b \neq a$, then $(a + x) + y = b + (x + y)$. This implies, $\{(a + x) + y\} \wedge b = \{b + (x + y)\} \wedge b$ or $0 = b$, a contradiction. Hence, a belongs to the left nucleus of A and since a generates A, the left nucleus of A is all of A. That is, A is associative.

Definition (2.13): If a is a positive atom we will denote the unique element u such that

$u + a = a + u = 0$ by $\bar{a}$, and call it a negative atom.

Lemma (2.7): If $\bar{a}$ is a negative atom, then $\bar{a}$ generates a semi group under addition.

Proof: Same as above proof.

Lemma (2.8): Let w $(a_1, a_2, a_3\ldots)$ be an element in the groupoid generated by the positive atoms $a_1, a_2, a_3\ldots$ under addition. If w contains m_i occurrences of a_i, i=1, 2, 3 … then $w = \vee_i m_i a_i$.

Proof: We note that the notation $m_i a_i$ is unambiguous. Since each atom generates a semi group.

The proof is by induction on the length of w as a groupoid word in the a_i. The statement is clearly true for w of length one. Let w $(a_1, a_2, a_3\ldots) = p\,(a_1, a_2, a_3\ldots) + q\,(a_1, a_2, a_3\ldots)$.

Hence $w = \vee_i p_i a_i + \vee_j q_j a_j$ by our induction hypothesis we will assume that in this expression for w, each join $\vee_i p_i a_i$, $\vee_j q_j a_j$ contains terms $p_k a_k$, $q_k a_k$ respectively, for each atom a_k which occurs in w, with $0a_k$ defined to be 0.

By distributivity of addition over join, $w = \vee_i (p_i a_i + \vee_j q_j a_j) = \vee_i \vee_j (p_i a_i + q_j a_j)$.

If i= j, then $p_i a_i + q_i a_i = (p_i + q_i)a_i$ since a_i generates a semi group.

If $i \neq j$, then $p_i a_i + q_j a_j = p_i a_i \vee q_j a_j$ since $p_i a_i \wedge q_j a_j = 0$.

Hence, $w = \{ \vee_i (p_i+q_i)a_i\} \vee x_1 \vee x_2 \vee \ldots$.. where each x_i is either some $p_i a_i$ or $q_j a_j$ and hence less than or equal to $\vee_i (p_i+q_i)a_i$. This completes the induction step and the lemma follows.

Lemma (2.9): L^+ is a commutative semi group.

Proof: This follows immediately from the preceding lemma.

Lemma (2.10): L- is a commutative semi group.

Lemma (2.11): Any element in lattice ordered loop L can be written as

(m1a1 + m2a2 +….) + (n1$\overline{a1}$ + n2$\overline{a2}$ +…….)

Proof: If x $\in$ L, then x = (x $\vee 0) + (x \wedge 0)$. **Since x** $\vee 0 \in$ **L+, x** $\wedge$ **0** $\in$ **L-, Lemma (2.11) follows the sequence of** preceding lemmas.

3. CONCLUSIONS

This study makes it possible to

- Initiate positive (negative) cone of a lattice ordered loop and to derive various properties of positive (negative) cone.
- Observe that any lattice ordered loop is determined to within isomorphism by its positive cone.
- Identify that in any lattice ordered loop the positive cone is invariant under all inner-automorphisms.
- Derive that a po-loop is directed if and only if every element is expressed as the difference of two positive elements.
- Set out the left nucleus of a loop.
- Recognize that a positive (negative) atom generates a semigroup under addition.
- Distinguish that a positive (negative) cone is a commutative semigroup.

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